Probability and Counting

**Probability**

**We are interested in measuring the likelihood, or probability, of events occurring.**

**For example, suppose we want to estimate likelihoods of former CST students falling into certain categories. Assume we have access to all student records.**

**1. What is the probability that a randomly-chosen MATH 3042 student got a mark of over 75% in MATH 2121?**

**2. What is the probability that a randomly-chosen MATH 3042 student got a mark of over 75% in MATH 2121 and MATH 1113?**

**3. What is the probability that a randomly-chosen MATH 3042 student who got a mark of over 75% in MATH 2121 also got a mark of over 75% in MATH 1113?**

**In this section, we will investigate the *probabilities* associated with certain *events* in a *random* *experiment*. But first, we need to define some terms carefully:**

**A *random experiment* is one in which we can neither predict nor control which one of the many possible outcomes will actually occur in a given instance.**

**Examples:**

**1. Rolling a fair 6-sided die.**

**Possible outcomes:**

**2. Testing 2 dongles from a shipment.**

**Possible outcomes:**

**3. Measuring the capacity of a flash drive labeled 16GB.**

**Possible outcomes:**

**The *sample space,* typically denoted *S,* is the set of all possible outcomes for a random experiment.**

**Examples:**

**1. Rolling a fair 6-sided die.**

**2. Testing 2 dongles from a shipment.**

**3. Measuring the capacity of a flash drive labelled 16GB.**

**An *event* is any subset of the sample space, i.e. a collection of outcomes from the sample space. [Note: This could include just one outcome, in which case we say that the event is *simple*.]**

**Examples:**

1. **Rolling a fair 6-sided die.**

1. **Testing 2 dongles from a shipment.**

1. **Measuring the capacity of a flash drive labeled 16GB.**

**P denotes probability. *A* denotes specific events. P(*A*) denotes the probability of event *A* occurring.**

**Relative frequency Approach to Probability**

**Conduct (or observe) an experiment a large number of times, and count the number of times that event *A* occurs. Then P(*A*) is estimated as follows:**

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**Classical Approach to Probability**

**Assume that a given experiment has n different simple events, each of which has an *equal chance* of occurring. If event A can happen s of these n ways, then**

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**Law of Large Numbers**

**As an experiment is repeated, the relative frequency probability of an event tends to approach the actual probability.**

**Example:**

**Suppose we wish to find the probability of rolling a 3 on a single die** **.**

**Solution (Relative Frequency Approach):**

**Conduct an experiment and roll a die "many" times and record the number of times that the number three occurs to estimate the probability.**

**We can simulate die-rolling in R.**

> dice10<-sample(1:6,10,replace=TRUE)

> dice10

[1] 1 5 6 2 3 1 1 3 2 1

> sum(dice10==3)

[1] 2

|  |  |  |
| --- | --- | --- |
| **Number of rolls** | **Number of 3’s Observed** | **Approximate Probability** |
| **10** |  |  |
| **100** | > dice100<-sample(1:6,100,replace=TRUE)  > sum(dice100==3)  [1] 18 |  |
| **1000** | > dice1000<-sample(1:6,1000,replace=TRUE)  > sum(dice1000==3)  [1] 162 |  |

**Solution (Classical Approach):**

**There is only one way you can roll a 3 and there are six possible and equally likely events in the sample space.**

**P(rolling a 3) = 1/6 ≈ 0.167**

**Notice in our experiment that our estimate approached the actual probability as we increased the number of trials.**

**Counting**

**Fundamental Counting Rule**

**If sets A1, A2, …, Ak contain, respectively n1, n2, … , nk elements, there are n1`× n2 × … × nk ways of choosing first and element of A1, then an element of A2, …, and finally an element of Ak.**

**Examples:   
1. There are 14 students in ELEX 5S, 17 students in COMP 4C, 18 students in COMP 4D, 10 students in ROBT 3A, and 12 students in ROBT 3B. In how many ways can we select a representative from each set?**

**2. Each byte contains 8 bits, how many possible distinct bytes are there?**

1. **What is the probability of getting 4 heads in 4 successive flips of a coin?**

**We will use the classical approach to probability along with the Fundamental Counting Rule:**

**According to the classical approach to probability:**

**P(4H) =**

**According to the Fundamental Counting Rule, the number of possible outcomes of 4 flips is:**

1. **What is the probability of getting 2 heads and 2 tails in any order on 4 successive flips of a coin?**
2. **What is the probability of getting a head and a 5 when tossing a coin and a die?**

**In this course we will be seeing several examples involving decks of cards.**

**A standard deck of card contains 52 cards. There are 13 cards of each suit: clubs, diamonds, hearts, and spades. The ranks of each suit are: 2, 3, 4, 5, 6, 7, 8, 9, 10, J(ack), Q(ueen), K(ing), A(ce).**

1. **Find the probability of drawing a red card or a queen in a single draw from a standard deck of 52 playing cards.**

**Factorials, Permutations, Combinations**

**Factorials are the product of successive whole numbers.**

**6! = 6 x 5 x 4 x 3 x 2 x 1 = 720**

**Factorial Rule**

**A collection of n different items can be arranged in order n! different ways.**

**Note: 0! = 1**

**Examples:**

1. **How many ways are there to arrange 100 students in a classroom containing 100 seats?**
2. **How many ways are there to fill the first row of a classroom (10 seats total) with students from a class of 100?**

**Often we got a product of a sequence of consecutive whole numbers, but the sequence didn't go all the way down to 1. How can we express our result from #2 in terms of factorials?**

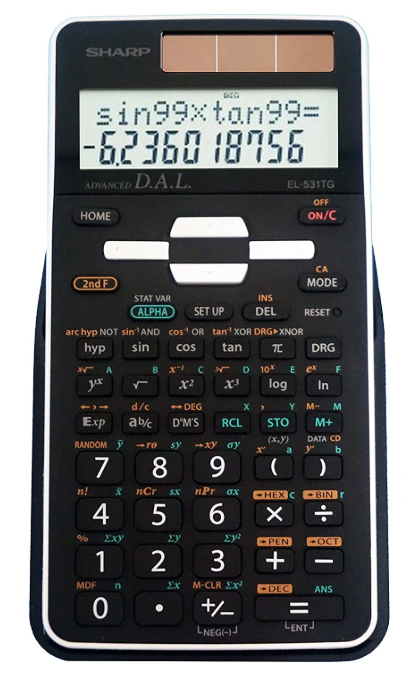
**This is an example of a *permutation of 10 objects from a set of 100.***

**Permutation Rule (When all items are different)**

**The number of permutations of r items selected from n available items (not allowing repetition) is**

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**On your calculator:**

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**Examples:**

1. **How many ways are there to get three diamonds in three draws from a standard deck of playing cards when each card is *not* replaced before the next is drawn? Note: “ace of diamonds, 4 of diamonds, 9 of diamonds” is considered a different way of getting 3 diamonds than “9 of diamonds, 4 of diamonds, ace of diamonds”.**
2. **The BCIT Mathematics Department has 20 instructors. A 5 person committee is needed to interview prospective new instructors. How many ways can the committee be constructed if the committee is to have officers (President, Vice-President, Secretary, Treasurer, and Observer)?**

**First we must determine if this is a permutation or combination situation.**

**Consider the following two committees:**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **President** | **Vice-President** | **Secretary** | **Treasurer** | **Observer** |
| **Paul** | **Mirela** | **Stephen** | **Bill** | **Erin** |
| **Erin** | **Stephen** | **Bill** | **Paul** | **Mirela** |

1. **For any data network, we wish to route the data packets through the shortest (“fastest”) possible path through the network. Suppose a network has 5 nodes, how many possible paths are there through the network assuming that each node is connected to all other nodes and that once a node is visited, it cannot be revisited?**
2. **Suppose you have an array that contains 7 distinct elements. How many possible sub-arrays containing 3 elements could you create from your original array?**
3. **How many ways are there to get three diamonds in three draws from a standard deck of playing cards when each card is *not* replaced before the next is drawn? This time, we will assume we sort our hand after we get our cards. So “ace of diamonds, 4 of diamonds, 9 of diamonds” is considered the same way of getting 3 diamonds than “9 of diamonds, 4 of diamonds, ace of diamonds”.**

**In the previous example, the order in which we dealt the cards was irrelevant. We needed to make sure we didn’t count the hands that contained the same set of cards.**

**Combinations Rule**

**The number of combinations of r items selected from n different items where order doesn't matter is**

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**To help determine whether this is a permutation or combination question, ask,**

**“Is a committee that consists of Paul, Mirela, Stephen, Erin, and Bill, different from a committee that consists of Bill, Stephen, Erin, Mirela, and Paul?”**

**The answer is no. So the order does not matter. This is a combination problem since the order in which the people are chosen is irrelevant.**

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**So there are 15,504 possible 5 person committees that can be created.**

**Examples:**

1. **The BCIT Mathematics Department has 20 instructors. If a 5 person committee is needed to interview prospective new instructors, how many ways can the committee be constructed?**

**These two committees contain the same 5 people but the committees themselves are different since different people hold the various offices. Thus “order” matters in this situation, so we must count the number of permutations.**

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**(Some scientific calculators have function keys for nPr and nCr, so that the expansion into factorials and the expansion of the factorials as done above would not be necessary.)**

1. **Find the probability of getting a spade flush (5 spades) in a 5 card poker hand.**

Probability Axioms

* **The probability of an impossible event is 0.**
* **The probability of an event that is certain to occur is 1.**
* **0 ≤ P(A) ≤ 1 for any event A.**

**Complementary Events**

**The complement of event A, denoted by , consists of all outcomes in which event A does NOT occur.**

**Eg – rolling a die.**

**A = rolling an even number**

** =**

**By definition of complement, we have the fact that**

**P(A) + P() =\_\_\_\_\_\_.**

Often times it is easier to calculate the probability of an event not happening than it is to calculate the probability that it does.

**Example: You roll 2 dice. What is the probability that the sum of your numbers is less than 12?**

**Birthday Problem**

**Find the probability that among ­­­\_\_­­­­­\_\_\_ people, at least two people have birthdays on the same date (but not necessarily the same year). Assume that all months and dates are equally likely, and ignore February 29th birthdays.**

**Compound Events**

**A compound event is any event combining two or more simple events.**

**Addition Rule**

**P(A ∪ B) = P(A) + P(B) - P(A ∩ B)**

* **where P(A ∪ B) denotes the probability that A occurs or B occurs or both A and B occur at the same time as an outcome in one experiment.**
* **where P(A ∩ B) denotes the probability that A and B both occur at the same time as an outcome in one experiment.**

**Definition:**

**Events A and B are mutually exclusive if they cannot occur simultaneously.**

**eg –**

**A=draw a black card**

**B=draw a heart**

**Whenever A and B cannot occur simultaneously, P(A ∩ B) = \_\_\_\_\_\_**

When events A and B are mutually exclusive, P(A ∩ B) = \_\_\_\_\_\_

and the addition rule becomes:

Examples:

1. When drawing the top card from a standard deck of cards, what is the probability of selecting a red card or an ace?



1. When drawing the top card from a standard deck of cards, what is the probability of selecting a 7 or an ace?

**Sometimes, the probability of an event B depends on whether or not a different event, A, has occurred.**

**Two events A and B are independent if the occurrence of one does not affect the probability of the occurrence of the other.**

**If A and B are not independent then they are said to be dependent.**

**Some Notation:**

P(B|A) represents the probability of event B occurring assuming that event A has already occurred.

**P(B|A) "The probability of B given A"**

Example: A: BCIT student is in CST

B: BCIT student knows how to create a linked list  
  
 How does P(B) compare to P(B|A)?

If A and B are independent, how does P(B) compare to P(B|A)?

In mathematics we have found that "or" or ∪ implies that we add. Similarly, there is a corresponding operation for "and" or ∩ situations. As you may have guessed, "and" implies that we multiply.

**Multiplication Rule**

**P(A ∩ B) = P(A) P(B) if A and B are independent.**

**P(A ∩ B) = P(A) P(B|A) if A and B are dependent.**

**Examples**

What is the probability of getting two heads in two flips of a fair coin?

  
  
  
  
  
Example: Two cards are drawn at random from an ordinary deck of cards. What is the probability of getting 2 aces if

1. **the first card is replaced before the second is drawn?**

1. **the first card is not replaced before the second is drawn?**

**We can rearrange the formula P(A ∩ B ) = P(A) P(B|A) to find**

**the conditional probability of B given A, which is the probability of event B occurring given that event A has already occurred.**

**Simply by rearranging the formula we have, **

**Bayesian Probabilities**

**Thomas Bayes (1702 - 1761) said that probabilities should be revised when we learn more about an event.**

**For instance, if we meet a BCIT student, we can estimate how likely it is that that student knows how to create a linked list. If we then find out that the student is in CST, we will update our estimate.**

**Revised probabilities are actually conditional probabilities, thus we have already examined much of the material in this section. We need only a bit more background.**

**Events A1 … Ak are said to be exhaustive if one or more of them must occur.**

**For example rolling a single die,**

**A1 = {1, 2}**

**A2 = {3, 4}**

**A3 = {5, 6}**

**A4 = {5}**

**When you roll the die one or more of events A1 … A4 must occur.**

**Note that to be exhaustive, you do not have to be mutually exclusive. Obviously, rolling a 5 belongs to two of the events A3 and A4.**

**Suppose we have the three events A1, A2, A­3 above. These events are both exhaustive and mutually exclusive. Suppose we have event B defined as B = {Even Rolls}.**

**Then, we have the following Venn Diagram**

**B**

**A3 ∩ B**

**A2 ∩ B**

**A1 ∩ B**

**A2**

**A3**

**A1**

**Notice how A1, A2, and A­3 completely fill the sample space.**

**We can see that the probability of rolling an even, P(B), can be found by adding the components of B found in each of the Ak regions.**

**P(B) = P(A1 ∩ B) + P(A2 ∩ B) + P(A3 ∩ B)**

**= P(A1)P(B|A1) + P(A2)P(B|A2) + P(A3)P(B|A3)**

**Rule of Total Probability**

**Suppose events A1, A2, … Ak are mutually exclusive and exhaustive; that is, exactly one of the events must occur. Then for any event B,**

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**Example:**

|  |  |  |
| --- | --- | --- |
| **Company** | **Market Share** | **Percent Defective** |
| **Intel** | **85.0%** | **1.0%** |
| **AMD** | **10.0%** | **3.0%** |
| **Other** | **5.0%** | **2.0%** |

1. **What is the probability of randomly selecting a processor that is defective?**

**Solution:**

|  |  |  |
| --- | --- | --- |
| **Company** | **Market Share** | **Percent Defective** |
| **Intel** | **85.0%** | **1.0%** |
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**Let:**

**B = defective chip**

**A1 = processor selected is Intel**

**A2 = processor selected is AMD**

**A3 = processor selected is Other**

**P(A1) = P(A2) = P(A3) =**

**P(B|A1) = P(B|A2) = P(B|A3) =**

**Tree Diagram**

In the previous example we wished to know the probability of choosing a defective processor. Suppose we select a defective processor and wish to know the probability that the processor was built by Intel. This is the type of problem that Thomas Bayes focused his attention.

**Non-Defective**

**Defective**

**Non-Defective**

**Defective**

**Non-Defective**

**Defective**

**0.98**

**0.97**

**0.99**

**0.02**

**0.03**

**0.01**

**0.05**

**0.10**

**0.85**

**A3**

**A2**

**A1**

**Using the Rule of Total Probability, Bayes developed what is now known as Bayes' Rule:**

**Bayes' Rule**

**If events A1, A2, …, Ak are mutually exclusive and exhaustive, then for any event B,**

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**where Ai can be any one of the events A1, A2, …, Ak.**

**Example (Continued).**

Suppose we select a defective processor and wish to know the probability that the processor was built by Intel.

**P(Intel | Defective) = P(A1|B)**

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